# Transverse momentum and energy correlations in the equilibrium system from high-energy nuclear collisions *

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The so-called $\Phi$ parameter, which measures the transverse momentum or energy correlations (fluctuations) in high-energy collisions independently of the particle multiplicity, is computed for the equilibrium ideal gas. As expected, $\Phi$ vanishes for the particles obeying Boltzmann statistics but is finite for the quantum particles, positive for bosons and negative for fermions. $\Phi_{p_\perp}$, which is found for the pions gas, significantly exceeds the value of $\Phi_{p_\perp}$ measured by the NA49 experiment. The discrepancy is discussed.

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There is a variety of correlations observed in proton–proton or proton–antiproton interactions at high energies. In particular, it has been found that the average particle transverse momentum depends on the particle multiplicity in a given collision [1,2]. These correlations should be also present in nucleus–nucleus (A–A) collisions if such a collision is a superposition of nucleon–nucleon (N–N) interactions. However, there is no straightforward method to observe them since the final state particles in A–A collisions originate from the various N–N interactions while the correlated particles come only from the same N–N interaction.

In our earlier paper [3] we have introduced a rather tricky quantity (later called $\Phi$) which appears to be sensitive to the correlations independently of the particle multiplicity. If the A–A collision is a superposition of N–N interactions with no secondary collisions, the value of $\Phi$ is exactly the same for the N–N and A–A case. If the secondary interactions play an important role in nucleus–nucleus collisions, the correlation of interest is reduced [4] and in the limiting case, when the final sate particles are independent from each other, $\Phi$ equals zero. The method developed in [3] has been recently applied to the experimental data and it has been found [5,6] that the correlation, which is present in N–N collisions, survives in proton–nucleus ones but is significantly reduced in the central Pb–Pb collisions. This result appears to be a very restrictive test of the models describing the N–N and A–A collisions. For example, the so-called random walk model [8] is ruled out because it has been shown to produce, in contrast to the data, the stronger correlations in A–A than N–N case [7].

Reduction of the correlations measured by $\Phi$ in the central A–A collisions is naturally associated with the evolution of the system produced in these collisions towards the thermodynamical equilibrium. However, it has been correctly observed in [7] that in the thermodynamical equilibrium the rudimentary correlation should be present. Our aim is to substantiate this observation.

Let us first introduce the correlation (or fluctuation) measure $\Phi_x$, where $x$ is a single particle characteristics such as the particle energy or transverse momentum. We define the variable $z_x \stackrel{\rm def}{=} x - \overline{x}$ with the overline denoting averaging over a single particle inclusive distribution. As seen $\overline{z}_x = 0$. We now introduce the variable $Z$, which is a multiparticle analog of $z$, defined as $Z_x \stackrel{\rm def}{=} \sum_{i=1}^{N}(x_i - \overline{x})$, where the summation runs over particles from a given event i.e. the particles which are produced in the collision. One observes that $\langle Z_x \rangle = 0$, where $\langle ... \rangle$ represents averaging over events (collisions). Finally, we define the measure $\Phi_x$ in the following way

$$\Phi_x \stackrel{\rm def}{=} \sqrt{\frac{\langle Z_x^2 \rangle}{\langle N \rangle}} - \sqrt{\overline{z_x^2}} \, . \tag{1}$$

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Our purpose is to calculate $\Phi_x$ in the ideal quantum gas. At the beginning we identify $x$ with the particle energy $E$ and then we consider the particle transverse momentum $p_\perp$.

One immediately finds that

$$\overline{z_E^2} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \, (E - \overline{E})^2 \frac{1}{\lambda^{-1} e^{\beta E} \pm 1} \; , \tag{2}$$

where the single particle average energy is

$$\overline{E} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \, \frac{E}{\lambda^{-1} e^{\beta E} \pm 1}$$

while $\rho$ equals

$$\rho = \int \frac{d^3p}{(2\pi)^3} \, \frac{1}{\lambda^{-1} e^{\beta E} \pm 1} \; ; \tag{3}$$

$\beta \equiv T^{-1}$ is the inverse temperature; $\lambda \equiv e^{\beta\mu}$ denotes the fugacity and $\mu$ the chemical potential; $E \equiv \sqrt{m^2 + \mathbf{p}^2}$ with $m$ being the particle mass and $\mathbf{p}$ its momentum; the upper sign is for fermions while the lower one for bosons. It is worth noting that the result (2) does not depend on the number of the particle internal degrees of freedom.

Since $Z_E = U - N\overline{E}$, where $U$ is the system energy, $\langle Z_E^2 \rangle$ is computed as

$$\langle Z_E^2 \rangle = \frac{1}{Xi} \left[ \frac{\partial^2}{\partial\beta^2} + 2\overline{E}\, \lambda \frac{\partial^2}{\partial\beta\, \partial\lambda} + \overline{E}^2 \lambda \frac{\partial}{\partial\lambda} \left( \lambda \frac{\partial}{\partial\lambda} \right) \right] \Xi(V,T,\lambda) \; ,$$

where $\Xi(V,T,\lambda)$ is the grand canonical partition function [9] defined as

$$\Xi(V,T,\lambda) = \sum_N \sum_\alpha \lambda^N e^{-\beta U_\alpha} \; ,$$

with $V$ denoting the system volume and the index $\alpha$ numerating the system quantum states. As well known [9], the grand canonical partition function of the quantum ideal gas equals

$$\ln \Xi(V,T,\lambda) = \pm g\, V \int \frac{d^3p}{(2\pi)^3} \, \ln\bigl[1 \pm \lambda\, e^{-\beta E}\bigr] \; ,$$

with $g$ being the number of the particle internal degrees of freedom. Consequently,

$$\frac{\langle Z_E^2 \rangle}{\langle N \rangle} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \, (E - \overline{E})^2 \frac{\lambda^{-1} e^{\beta E}}{(\lambda^{-1} e^{\beta E} \pm 1)^2} \; . \tag{4}$$

As previously the result is independent of $g$. One observes that

$$\frac{\langle Z_E^2 \rangle}{\langle N \rangle} < \overline{z_E^2} \qquad \text{and} \qquad \Phi_E < 0$$

for fermions,

$$\frac{\langle Z_E^2 \rangle}{\langle N \rangle} > \overline{z_E^2} \qquad \text{and} \qquad \Phi_E > 0$$

for bosons and

$$\frac{\langle Z_E^2 \rangle}{\langle N \rangle} = \overline{z_E^2} \qquad \text{and} \qquad \Phi_E = 0$$

in the classical limit i.e. when $\lambda^{-1} \gg 1$.

In the case of massless particles with vanishing chemical potential (which corresponds to $\lambda = 1$), one finds $\Phi_E$ analytically. Namely,

$$\rho = \frac{\zeta(3)}{\pi^2} \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} T^3 \cong \begin{pmatrix} 0.09 \\ 0.12 \end{pmatrix} T^3$$

and

$$\overline{E} = \frac{\pi^4}{30\zeta(3)} \begin{pmatrix} 7/6 \\ 1 \end{pmatrix} T^3 \cong \begin{pmatrix} 3.15 \\ 2.70 \end{pmatrix} T \ ,$$

where $\zeta(x)$ is the Riemann zeta function ($\zeta(3) \cong 1.202$, $\zeta(5) \cong 1.037$); the upper case is for fermions and the lower one for bosons. Further one computes

$$\overline{z_E^2} = \frac{12\zeta(5)}{\zeta(3)} \begin{pmatrix} 5/4 \\ 1 \end{pmatrix} T^2 - \overline{E}^2 \cong \begin{pmatrix} 3.01 \\ 3.06 \end{pmatrix} T^2 \ , \tag{5}$$

$$\frac{\langle Z_E^2 \rangle}{\langle N \rangle} = \frac{1}{\rho} \left[ \frac{2\pi^2}{15} \begin{pmatrix} 7/8 \\ 1 \end{pmatrix} T^5 - \frac{6\zeta(3)}{\pi^2} \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} \overline{E}\, T^4 + \frac{1}{6} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \overline{E}^2\, T^3 \right] \cong \begin{pmatrix} 2.77 \\ 4.59 \end{pmatrix} T^2 \ , \tag{6}$$

which finally give

$$\Phi_E \cong \begin{pmatrix} -0.07 \\ 0.40 \end{pmatrix} T \ .$$

When the system is composed of the equal mass fermions and bosons with the numbers of the internal degrees of freedom $g_f$ and, respectively, $g_b$, the analogs of the formulas (2) and (4) read

$$\overline{z_E^2} = \frac{1}{g_f \rho_f + g_b \rho_b} \int \frac{d^3p}{(2\pi)^3} \, (E - \overline{E})^2 \left[ \frac{g_f}{\lambda_f^{-1} e^{\beta E} + 1} + \frac{g_b}{\lambda_b^{-1} e^{\beta E} - 1} \right] , \tag{7}$$

$$\frac{\langle Z_E^2 \rangle}{\langle N \rangle} = \frac{1}{g_f \rho_f + g_b \rho_b} \int \frac{d^3p}{(2\pi)^3} \, (E - \overline{E})^2 \left[ \frac{g_f \lambda_f^{-1} e^{\beta E}}{(\lambda_f^{-1} e^{\beta E} + 1)^2} + \frac{g_b \lambda_b^{-1} e^{\beta E}}{(\lambda_b^{-1} e^{\beta E} - 1)^2} \right] . \tag{8}$$

Using eqs. (5, 6) and (7, 8) one easily computes $\Phi_E$ for the baryonless quark-gluon plasma of two flavours where $g_f = 24$ and $g_b = 16$; $\Phi_E \cong 0.17\, T$.

Eqs. (2, 4) can be used to get the measure $\Phi_{p_\perp}$ of the transverse momentum fluctuations. Since $p_\perp = p \sin\Theta$ with $p \equiv |\mathbf{p}|$ and $\Theta$ being the angle between $\mathbf{p}$ and the beam ($z$) axis one gets

$$\begin{aligned} \overline{z_{p_\perp}^2} &= \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \left( p_\perp - \overline{p}_\perp \right)^2 \frac{1}{\lambda^{-1} e^{\beta E} \pm 1} \\ &= \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \left( \frac{2}{3} p^2 - \frac{\pi}{2} \overline{p}_\perp \, p + \overline{p}_\perp^2 \right) \frac{1}{\lambda^{-1} e^{\beta E} \pm 1} \ , \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\langle Z_{p_\perp}^2 \rangle}{\langle N \rangle} &= \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \left( p_\perp - \overline{p}_\perp \right)^2 \frac{\lambda^{-1} e^{\beta E}}{(\lambda^{-1} e^{\beta E} \pm 1)^2} \\ &= \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \left( \frac{2}{3} p^2 - \frac{\pi}{2} \overline{p}_\perp \, p + \overline{p}_\perp^2 \right) \frac{\lambda^{-1} e^{\beta E}}{(\lambda^{-1} e^{\beta E} \pm 1)^2} \ , \end{aligned} \tag{10}$$

where

$$\overline{p}_\perp = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \, \frac{p_\perp}{\lambda^{-1} e^{\beta E} \pm 1} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \, \frac{p \sin\Theta}{\lambda^{-1} e^{\beta E} \pm 1} = \frac{\pi}{4\rho} \int \frac{d^3p}{(2\pi)^3} \, \frac{p}{\lambda^{-1} e^{\beta E} \pm 1} \ . \tag{11}$$

Let us observe that $\Phi_{p_\perp}$ is invariant under the Lorentz boosts along the beam axis. This is evident when eqs. (9, 10) and (3, 11) are written in the from which reveals the transformation properties. We consider as an example the average transverse momentum which can be expressed as

$$\overline{p}_\perp = \frac{1}{\rho} \int \frac{d^2p_\perp dp_\parallel}{(2\pi)^3} \, \frac{p_\perp}{\lambda^{-1} e^{\beta u^\nu p_\nu} \pm 1} \ ,$$

with

$$\rho = \int \frac{d^2p_\perp dp_\parallel}{(2\pi)^3} \, \frac{1}{\lambda^{-1} e^{\beta u^\nu p_\nu} \pm 1} \; ,$$

where $u^\nu$ is the four-velocity which determines the reference frame; $u^\nu = (1,0,0,0)$ corresponds to the thermostat rest frame. One sees that the two integrals, which determine $\overline{p}_\perp$, are both frame dependent due to the presence of $dp_\parallel$. However, the dependence cancels out in the ratio of the integrals. Analogously one shows that $\Phi_{p_\perp}$ is invariant.

We consider again the case of massless particles with vanishing chemical potential. As previously the calculations are performed in the thermostat rest frame. Then,

$$\overline{p}_\perp = \frac{\pi^5}{120\zeta(3)} \begin{pmatrix} 7/6 \\ 1 \end{pmatrix} T^3 \cong \begin{pmatrix} 2.48 \\ 2.12 \end{pmatrix} T \; ,$$

$$\overline{z^2_{p_\perp}} = \frac{8\zeta(5)}{\zeta(3)} \begin{pmatrix} 5/4 \\ 1 \end{pmatrix} T^2 - \overline{p}^2_\perp \cong \begin{pmatrix} 2.50 \\ 2.40 \end{pmatrix} T^2 \; ,$$

$$\frac{\langle Z^2_{p_\perp} \rangle}{\langle N \rangle} = \frac{1}{\rho} \left[ \frac{4\pi^2}{45} \begin{pmatrix} 7/8 \\ 1 \end{pmatrix} T^5 - \frac{3\zeta(3)}{2\pi} \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} \overline{p}_\perp \, T^4 + \frac{1}{6} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \overline{p}^2_\perp \, T^3 \right] \cong \begin{pmatrix} 2.33 \\ 3.37 \end{pmatrix} T^2 \; ,$$

which provide

$$\Phi_{p_\perp} \cong \begin{pmatrix} -0.05 \\ 0.29 \end{pmatrix} T \; .$$

When the gas particles are massive and/or the chemical potential is finite, the correlation measure $\Phi_{p_\perp}$ can be numerically computed directly from eqs. (9) and (10). In Figs. 1 and 2 we show $\Phi_{p_\perp}$ as function of $T$ and $\mu$ for the pion gas. The pions are, of course, massive with $m = 140$ MeV. One sees that the presence of the finite mass reduces the correlation measure $\Phi_{p_\perp}$ when compared to the massless case.

As already mentioned $\Phi_{p_\perp}$ has been experimentally measured in the central Pb–Pb collisions by the NA49 collaboration. The result is $\Phi_{p_\perp} = 0.7 \pm 0.5$ MeV [5]. If we identify the system freeze-out temperature with the slope parameter deduced from the pion transverse momentum distribution $T \cong 180$ MeV [10]. Then, the value of $\Phi_{p_\perp}$, which is read out from Fig. 1 for $T = 180$ MeV and $\mu = 0$, equals 15 MeV and drastically exceeds the experimental value. The temperature is significantly reduced if the transverse hydrodynamic expansion is taken into account. The freeze-out temperature obtained by means of the simultaneous analysis of the single particle spectra and Bose-Einstein correlations is about 120 MeV [10]. The value of $\Phi_{p_\perp}$ for $T = 120$ MeV and $\mu = 0$ equals 6.5 MeV and is still much larger than the experimental value. Let us discuss this puzzling discrepancy.

$\Phi_{p_\perp}$ has been measured for pions which come form the limited phase-space region: $0.005 < p_T < 1.5$ GeV and $4 < y < 5.5$ [5]. However, it should not distort the value of $\Phi_{p_\perp}$ noticeably. First of all, one sees that the acceptance domain of $p_T$ covers the $p_T-$region which contributes to the integrals from eqs. (9) and (10). One notes that the average $p_T$ approximately equals $2T$. Secondly, we observe that the system longitudinal expansion influence the value of $\Phi_{p_\perp}$ insignificantly as long as the transverse momentum distribution weakly depends on the particle rapidities. Finally, one notes that the size of the longitudinal momentum domain does not matter very much for the value of $\Phi_{p_\perp}$. Therefore, we conclude that the finite acceptance of the NA49 measurement cannot be responsible for the discussed discrepancy.

We have considered three other ways to reconcile the experimental and theoretical values of $\Phi_{p_\perp}$.

- The pions are out of chemical equilibrium and the chemical potential is negative. It appears however that $\mu$ must acquire an unrealistically large negative value ($\mu = -\ 245$ MeV) to get $\Phi_{p_\perp} = 0.7$ MeV.

- A substantial fraction of the final state pions come from the hadron resonances. These pions do not ‘feel’ the Bose-Einstein statistics at freeze-out and should be treated as particles which obey the Boltzmann statistics. Then, they do not contribute to $\Phi_{p_\perp}$. Assuming that the fraction $k$ of the final state pions come from the resonances, $\Phi_{p_\perp}$ is reduced approximately by a factor $\sqrt{1-k}$. The value of $k$ must be again unrealistically large to reduce $\Phi_{p_\perp}$ sufficiently.

- The Coulomb repulsion among the like-sign pions is known to significantly diminish the bosonic correlations, see e.g. [11]. However, taking into account the electromagnetic interaction should not change the value of $\Phi_{p_\perp}$ noticeably. The point is that the Coulomb repulsion moderates the effect of boson statistics but the attraction among the unlike-sign pions generates the positive correlation of the similar size [11].

It is possible that the combination of the three effects considered above sufficiently reduces the theoretical value of $\Phi_{p_\perp}$. However, it is not a simple problem to perform a numerically reliable analysis. Therefore, we leave it for the future studies.

At the end let us comment on a somewhat paradoxical implication of our study. When the correlation measure $\Phi$ was introduced [3], we expected that the value of $\Phi$ would be smaller in A-A than in p-p interactions. We shared a rather common opinion that the correlations observed in p-p case were of a dynamical origin and tended to be washed out in A-A by rescatterings. The recently obtained experimental data [5,6] comply with our expectation. Indeed, $\Phi_{p_\perp} = 4.2 \pm 0.5$ MeV from p-p is about 6 times larger than that one from the central Pb-Pb collisions at the same energy 158 A $\cdot$ GeV [5,6]. However, we have now found that the equilibrium value of $\Phi_{p_\perp}$, which significantly exceeds $\Phi_{p_\perp}$ from Pb-Pb, is close to that one from p-p. This implies that even in p-p collisions the origin of the correlations is poorly understood. Therefore, a complete solution of the puzzle raised by our equilibrium calculation requires a better understanding of the effects which control the value of $\Phi$ at N-N level.

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## Figure Captions

**Fig. 1.** The correlation measure $\Phi_{p_\perp}$ as a function of temperature $T$ for four values of the chemical potential $\mu$. The most upper line is for $\mu = 70$ MeV, the second one for $\mu = 0$ etc.

**Fig. 2.** The correlation measure $\Phi_{p_\perp}$ as a function of chemical potential $\mu$ for four values of the temperature $T$. The most upper line is for $T = 200$ MeV, the second one for $T = 160$ MeV etc.